

# Breathing and randomly walking pulses in a semilinear Ginzburg-Landau system

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## Abstract

We consider a system consisting of the cubic complex Ginzburg-Landau equation which is linearly coupled to an additional linear equation. The model is known in the context of dual-core nonlinear optical fibers with one active and one passive cores. By means of systematic simulations, we find new types of stable localized excitations, which exist in the system in addition to the earlier found stationary pulses. The new localized excitations include pulses existing on top of a small-amplitude background (that may be regular or chaotic) *above* the threshold of instability of the zero solution, and breathers into which stationary pulses are transformed through a Hopf bifurcation below the aforementioned threshold. A sharp border between stable stationary pulses and breathers, which precludes their coexistence, is identified. Stable bound states of two breathers with a phase shift  $\pi/2$  between their internal vibrations are found too. Above the threshold, the pulse is standing if the background oscillations are regular; if the background is chaotic, the pulse is randomly walking. With the increase of the system's size, additional randomly walking pulses are spontaneously generated. The random walk of different pulses in a multi-pulse state is partly synchronized due to their mutual repulsion. At a large overcriticality, the multi-pulse state goes over into a spatiotemporal chaos.

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## I. INTRODUCTION

The important role played by localized pulses (sometimes called “autosolitons” [1]) in models of the Ginzburg-Landau (GL) type [2] is well known. They find applications to plasmas and plasma-like media [1], [3]- [5], nonequilibrium semiconductors [1], hydrodynamics (Poiseuille flow [6], electrohydrodynamic convection [7]), nonlinear optics [8]- [12] , etc.

The simplest model which gives rise to localized pulses is the complex cubic GL equation,

$$u_t = \gamma_0 u - (g - i\sigma) |u|^2 u + (\gamma_1 + i\gamma_2) u_{xx}, \quad (1)$$

where  $\gamma_0$ ,  $g$ , and  $\gamma_1$  are positive coefficients of the linear gain, nonlinear losses, and effective diffusion (dispersive losses), while  $\sigma$  and  $\gamma_2$  (which may have either sign) account for the nonlinear frequency shift and dispersion, respectively. Equation (1) always has a single exact solitary-pulse (SP) solution [6,3], which is, however, unstable, as its background, i.e., the trivial solution  $u = 0$ , is obviously unstable due to the presence of the linear gain.

The simplest possibility to modify the model so that to produce *stable* SPs is to convert it into the quintic GL equation [13], setting  $\gamma_0 < 0$  and  $g < 0$ , and adding a nonlinear quintic dissipative term in order to provide for the overall stabilization. In the quintic GL equation, the trivial solution is stable since  $\gamma_0 < 0$ , and two solitary-pulse solutions, one unstable and one stable, may coexist at fixed values of parameters [14].

In this connection, it is necessary to mention a modification of the usual cubic GL equation proposed by Schöpf and Kramer [15] (see also the works [16]), in which  $\gamma_0 = 1$  and  $g$  is negative, while no higher-order stabilizing nonlinearity is added. The corresponding model lacks global stability, as a very large perturbation with a small wavenumber will provoke blowup. Nevertheless, numerical simulations have shown that finite-amplitude solutions in the form of both chaotic and stationary arrays of pulses may exist in this model, due to the stabilizing effect exerted by the dispersion and diffusion.

The quintic GL equation is, generally, a phenomenological model, as the derivation from the first principles can scarcely stop at the fifth-order nonlinearity (provided that it is a crucially important term rather than a small correction). Although this model can sometimes describe experimentally observed

waves with a fairly high accuracy, an example being the so-called dispersive chaos in the traveling-wave convection in binary fluids [17], it is desirable to find more models that allow for the existence of stable SPs and can be derived from the first principles. A possibility is to consider a model of a dual-core nonlinear optical fiber, in which the linear gain, dispersion, effective diffusion (spectral filtering), and Kerr (nondissipative cubic) nonlinearity are present in one (active) core, while the other (passive) one, linearly coupled to the active core, has only linear loss [8,9]. Thus, the model consists of a cubic GL equation linearly coupled to the second, purely linear, ordinary differential equation for the local amplitudes  $u$  and  $v$  of the waves in the two subsystems:

$$\begin{aligned} u_t &= \gamma_0 u + i\sigma|u|^2 u + (\gamma_1 + i\gamma_2)u_{xx} + iv, \\ v_t &= -\Gamma v + iu \end{aligned} \tag{2}$$

(note that, in the application to optical fibers, the evolutionary variable  $t$  is actually the distance along the fiber, while  $x$  is the so-called local time [18]). The parameters  $\gamma_0$ ,  $\gamma_1$ ,  $\gamma_2$  and  $\sigma$  of the active subsystem have the same meaning as in the cubic GL equation (1), there is no saturating nonlinear term,  $\Gamma$  is the loss factor in the passive subsystem, and the coefficient of the linear coupling between the subsystems is set equal to 1.

Note that an essentially more general version of the model, with the dispersion, Kerr, and filtering terms present in the equation for the passive core was also studied in detail [8,10]. However, both physical arguments and numerical results presented in [9] clearly demonstrate that the version of the model with the very simple form of the additional equation displayed above is quite sufficient to grasp all the essential dynamical features of the dual-core optical fiber with one active and one passive cores [9].

Although the model (2), as well as those considered in refs. [15] and [16], has only cubic nonlinearity, it is very different from them, having another stabilization mechanism. In particular, it has been demonstrated analytically and numerically that there is no global instability in the present model (i.e., it never gives rise to blowup) [8,9].

Parameters in eqs. (2) can be chosen so that to provide for the stability of the zero solution, which lends localized pulses, if any, a chance to be stable [8–10,22], see below. Then, two *exact* zero-velocity SP solutions to eqs. (2) can be found, following the pattern of the exact solution to the cubic GL equation,

in the form

$$u = u_0[\text{sech}(kx)]^{(1+i\mu)} e^{i\omega t}, \quad v = v_0[\text{sech}(kx)]^{(1+i\mu)} e^{i\omega t} \quad (3)$$

[9]. The two exact solutions have different values of  $u_0, v_0, k, \mu$  and  $\omega$ . Direct numerical simulations have clearly shown that one of the two exact SPs may be stable [9,10]. The simulations have revealed nontrivial stability borders for this SP in the parameter space of the model (“nontrivial” implies that the stability borders are different from stability conditions for the zero background). Interactions between stable SPs were also simulated in detail in this model [10]; in particular, it has been found that stable bound states of two pulses exist. Three-pulse states exist too, but they are unstable against symmetry-breaking perturbations.

The objective of this work is to search for stable localized states in the model (2) beyond instability borders of the stationary pulses. In section 2, we demonstrate that effectively stable pulses and sets of pulses exist in the region where the zero background is *unstable*. It should be mentioned that a possibility for a pulse to survive above the background-instability threshold is known in a GL model of another type [19], but in that case the stabilization is provided by the fact that the pulse is moving, due an extra symmetry-breaking term added to the GL equation, and, in a system with periodic boundary conditions, it may perform a round trip quickly enough to suppress the growing local perturbations. In our model, the situation is different: immediately above the background-instability threshold, we observe stable pulses resting on top of a background standing-wave pattern with a small amplitude. With further increase of the overcriticality, the background pattern keeps a rather small amplitude but becomes chaotic. In the course of this transition, SPs remain stable, although they become *randomly walking* ones, rather than keeping zero velocity. While we did not try to elaborate in detail how the randomization of the background gives rise to the random walk of the pulse, the effect seems to be quite similar to the Brownian motion, resembling the *Gordon-Haus effect* (random walk of solitons interacting with random noise) in optical fibers [18]. Moreover, additional pulses are spontaneously generated at larger values of the overcriticality, and spectacular correlations in the random walk of far separated pulses are observed in this case, due to their mutual repulsion.

Another finding, presented in section 3, is that one of generic modes of the instability of stationary

SPs (in the case when the zero background is stable) does not destroy them, but instead transforms them into stable *breathers* (vibrating SPs). Stable bound states of two breathers were found too, while bound states with a larger number of breathers were not found. All these types of stable nonstationary SPs are fairly novel to the present model, and may be of considerable interest for the general analysis of models of the GL type.

## II. SOLITARY PULSES ABOVE THE BACKGROUND-INSTABILITY THRESHOLD

### A. Stability of the zero background

As it was explained above, the first necessary condition for the unequivocal stability of SPs is stability of the zero solution. To investigate this issue, one should linearize eqs. (2) and look for a perturbation eigenmode  $\sim \exp(\lambda t + i q x)$ , where  $q$  is an arbitrary real perturbation wavenumber, and  $\lambda$  is the corresponding instability growth rate. This yields a dispersion equation for  $\lambda(q)$  [8–10,22]:

$$\lambda^2 + (\Gamma - \gamma_0 + \gamma_1 q^2 + i \gamma_2 q^2) \lambda + \Gamma(-\gamma_0 + \gamma_1 q^2 + i \gamma_2 q^2) = 0. \quad (4)$$

Onset of instability takes place when  $\text{Re } \lambda$  changes its sign at some (critical) value of  $q$ . As it follows from eq. (4), this critical condition gives rise to an equation for the critical value:

$$\frac{\gamma_2^2 q^4 \Gamma(-\gamma_0 + \gamma_1 q^2)}{(\Gamma - \gamma_0 + \gamma_1 q^2)^2} + \Gamma(-\gamma_0 + \gamma_1 q^2) + 1 = 0. \quad (5)$$

In the further analysis, we will display typical numerical results obtained at fixed values

$$\sigma = 1, \gamma_2 = 5, \gamma_1 = 0.9, \quad (6)$$

while  $\gamma_0$  will be varied at some fixed values of  $\Gamma$ . Note that the variation of the gain parameter  $\gamma_0$  is the most physically meaningful way to scan the results, as it can be easily changed in the optical experiment by adjusting the pump power, while the other parameters are fixed for a given experimental setup.

Figure 1 displays the line  $\gamma_0(q)$ , as obtained from eq. (5) (*neutral stability curve*) at the fixed values (6) and  $\Gamma = 1.35$ . Above the line, one has  $\text{Re } \lambda(q) > 0$  with  $\text{Im } \lambda(q) \neq 0$ , i.e., oscillatory instability takes place. The curve has a minimum  $(\gamma_0)_{\min} = 0.527$  at  $q = 0.54$ . That is, the zero solution is stable

if  $\gamma_0 \leq (\gamma_0)_{\min}$ , and when the critical gain  $(\gamma_0)_{\min}$  is reached, the zero background becomes unstable against finite-wavenumber perturbations. In view of the importance of the critical value of the gain, in fig. 2 we display it as a function of  $\Gamma$ . The cusp at  $\Gamma = 1.82$  is related to a switch of the instability mode at this point: at  $\Gamma < 1.82$ , the finite-wavenumber instability takes place, while at  $\Gamma > 1.82$  the value  $\gamma_0 = (\gamma_0)_{\min}$  is attained at  $q = 0$ .

## B. Standing and walking solitary pulses

Simulations of the dynamics of SPs were performed by means of a pseudospectral method, assuming periodic boundary condition in  $x$ , with the period 60 or 120 (in some cases, the period was 480, see below). As the initial configuration, we took the exact SP solutions (3), borrowing expressions for its parameters from [9]. In those cases when SPs were found to be stable in the simulations reported in Ref. [9], we also saw that they were stable. In this section, we focus on a possibility of existence of robust pulses *above* the critical gain  $(\gamma_0)_{\min}$ , when it appears that SPs cannot be stable.

Figure 3 displays SP produced by the simulations at  $\gamma_0 = 0.54$  and  $\Gamma = 1.35$ . As it is seen from fig. 1, this point is located slightly above the neutral stability curve, so that the corresponding overcriticality parameter is  $\epsilon \equiv [\gamma_0 - (\gamma_0)_{\min}] / (\gamma_0)_{\min} \approx 0.025$ . As is seen in fig. 3a, the weak background instability generates a small-amplitude standing-wave pattern, the stable localized pulse existing on top of it. To further illustrate the properties of this solution, in fig. 3b we specially display the time dependence of  $\text{Re } u$  at two points: in the center of the localized pulse, and in the background. Both dependences are periodic, but their frequencies are very different (the background oscillations are much slower), i.e., SP and the background are *not* synchronized.

The background oscillations are regular at  $\gamma_0 = 0.54$ ; however, they become *chaotic* for  $\gamma_0 > 0.55$ , at the same value  $\Gamma = 1.35$ . We use the term "chaotic", since the time evolution of the background is irregular and has no periodicity. Figure 4a displays a localized pulse existing on top of the chaotic background at  $\gamma_0 = 0.61$ , the corresponding overcriticality being  $\epsilon = 0.157$ . Figure 4(b) displays the time evolution of  $|u(x, t)|$ . Interaction of the localized pulse with the chaotic background apparently gives rise to a *random walk* of the pulse; however, the pulse is not destroyed. The temporal evolution

of the SP's central coordinate  $X_p$  is shown in fig. 5a. To analyze the character of the random walk of the localized pulse, we display in fig. 5(b) the time-averaged squared displacement  $\langle (\Delta X_p)^2 \rangle$ , where  $\Delta X_p \equiv \Delta X_p(t + \Delta t) - \Delta X_p(t)$ , vs. the temporal interval  $\Delta t$ . As it is obvious from fig. 5b, the random walk approximately obeys the diffusive law,  $\langle (\Delta X_p)^2 \rangle \sim \Delta t$ .

*Spontaneous formation* of new pulses takes place for larger  $\gamma_0$ . Note that spontaneous generation of effectively stable pulses by the unstable background was observed in simulations of another GL model in Ref. [19]. It is not quite clear whether there is a definite border for the spontaneous generation of SPs from the chaotic background. The pattern shown in fig. 6 was generated from a nearly zero initial state (transient evolution is not shown) at  $\gamma_0 = 0.64$ , corresponding to  $\epsilon = 0.214$ . Three localized pulses are spontaneously generated in this case. The background is chaotic, and the three SPs are not identical. It is clearly observed that the pulses repel each other, so that they keep nearly equal separations between themselves (recall we simulated the system (2) with periodic boundary conditions). Figure 6b displays the time evolution of  $|u(x, t)|$ . A noteworthy feature of the established dynamical state is that the random walk of all the three pulses is highly synchronized, so that the pulses keep nearly equal separations between themselves in the course of the random walk. The number of the spontaneously generated pulses grows with the system's size. In fig. 7 we show the established state at the same values of the parameters as in fig. 6, but with the size of the system 480 instead of 120. The separations between SPs remain nearly equal, while the synchronization of the random walk of SPs is weaker in a large system with the chaotic background. The synchronization does take place between adjacent pulses, but the full global synchronization does not occur, as is seen in fig. 7.

The number of the pulses in an established state increases with  $\gamma_0$  but it is not a uniquely defined function of the system's parameters and size. More detailed simulations (not shown here) demonstrate that extra pulses can be added, and some may be removed, i.e., their exact number depends on the way the state was made.

Finally, at large values of the overcriticality, many pulses are created, and the system goes over into a more irregular "turbulent" state (in which there is no spatially regular structure of localized pulses). The transition occurs close to at  $\gamma_0 = 1.0$ ; however, this depends on the number of pulses. Examples are

shown in fig. 8 for  $\gamma_0 = 1.05$  (a) and  $\gamma_0 = 2$  (b) (again, with  $\Gamma = 1.35$ ). Note that these states have been obtained starting from nearly-zero initial conditions. Generation of pulse sets and the turbulent behavior are seen in these pictures. The time evolution is very fast, and the final number of pulses is larger for  $\gamma_0 = 2$ .

### III. BREATHERS AND THEIR BOUND STATES

The exact pulse solutions to eqs. (2) found in [9] are stable only in a part of the parameter region where they exist [9,11]. Besides the loss of the stability of the zero background, other, more nontrivial destabilization mechanisms occur as well [9]. In particular, it seems very plausible that the stationary pulse can be destabilized through a Hopf bifurcation.

In fig. 9, we show a stable breather replacing the stationary pulse at  $\gamma_0 = 0.18$  and  $\Gamma = 0.2$ . Figure 9a displays the time evolution of the profile  $|u(x, t)|$ , and fig. 9(b) displays the time evolution of  $|u(x_p, t)|$  at the peak position. Limit-cycle intrinsic oscillations of the pulse are clearly seen. Breathing localized pulses were earlier found in a reaction-diffusion model [20] and in the quintic GL equation [21] (note that the breathing solution in fig. 9a has a form similar to the well-known two-soliton solution of the nonlinear Schrödinger equation,  $u = 4\eta[\cosh(3kx) + 3 \exp(4i\eta^2 t) \cosh(kx)]/[\cosh(4kx) + 4 \cosh(2kx) + 3 \cos(4\eta^2 t)]$ ). To present a whole family of the breathers, in fig. 10a we show the amplitude of the oscillations of  $|u(x_p, t)|$  at the breather's central point as a function of  $\gamma_0$  for  $\Gamma = 0.2$ . At the value  $\gamma_0 \approx 0.156$ , at which the oscillation amplitude vanishes, the breather changes into the stationary SP. Figure 10b displays the square amplitude  $A^2$  as a function of  $\gamma_0$  near the critical point. The linear dependence of  $A^2$  on  $\gamma_0$  implies  $A \propto \sqrt{\gamma_0 - \gamma_{0c}}$  near the critical point  $\gamma_{0c}$ , that is, the bifurcation is the supercritical Hopf bifurcation.

If the gain parameter  $\gamma_0$  becomes too small, no nontrivial solution can exist in the system. Thus, there is a minimum value of  $\gamma_0$  confining the existence region of the localized pulses. Collecting results from many runs of the simulations performed at different values of  $\gamma_0$  and  $\Gamma$ , while the other parameters took the fixed values (6), in fig. 11 we have drawn the phase diagrams of the localized solutions existing *below* the threshold of the zero-background's instability. The zero background is stable beneath the line marked by squares, the breathing pulses change into stationary ones at the line marked by crosses, and



the stationary pulses are changed by the zero solution at the line marked by diamonds. In accord with what was said above, the border between the stationary pulses and breathers (crosses in fig. 11) is *sharp*, i.e., there is no overlapping between them. That is a natural consequence of the *supercritical* character of the Hopf bifurcation (see above), which makes coexistence of stable stationary and oscillating solutions very implausible.

The Hopf-bifurcation line collides with the zero-background's instability line near  $\Gamma = 0.3$  and  $\gamma_0 = 0.27$ . For  $\Gamma > 0.3$ , stable breathing pulses do not exist. We stress that, unlike the effectively stable nonbreathing pulses considered in section 2, we have *not* observed stable breathing pulses on top of the small-amplitude background above the zero-background's instability threshold in the region  $\Gamma < 0.3$ .

Bound states of stable localized pulses also play an important role in the dynamics of GL-type models (see, e.g., [5] and [11]). Our simulations have revealed a possibility of forming a stable bound state of two breathers. Figure 12 displays a bound state found at  $\gamma_0 = 0.18$  and  $\Gamma = 0.2$ . A noteworthy feature of this bound state is that the phase shift between internal vibrations in the two breathers is  $\pi/2$ , i.e., one breather has a maximum amplitude when the amplitude of its mate is minimal, and vice versa. Attempts to generate a bound state of three breathers have failed. In this connection, it is relevant to mention that fully stable bound states of two stationary pulses can be readily generated in the same model, while complexes consisting of three pulses exist, but they are destroyed by perturbations breaking their symmetry [11].

#### IV. CONCLUSION

In this work, we have considered a model based on the cubic complex Ginzburg-Landau equation which is linearly coupled to an additional linear equation, which was introduced in [9] as a simplification of a more general model originally proposed in [8]. The model describes a dual-core nonlinear optical fiber with one active and one passive core. By means of systematic simulations, we have found new types of stable localized excitations in this model, existing in addition to the earlier found stationary pulses. The newly found excitations include pulses existing on top of a small-amplitude background (which is regular or chaotic) above the threshold of instability of the zero solution, and breathers into which stationary

pulses are transformed by a Hopf bifurcation below the zero-solution instability threshold. A sharp border between the stable stationary pulses and breathers was identified. Stable bound states of two breathers with a phase shift  $\pi/2$  between their internal vibrations have been found too. Above the threshold, the pulses are, respectively, standing if the background oscillations are regular, and randomly walking if the background is chaotic. With the increase of the system's size, additional randomly walking pulses are spontaneously generated. The random walk in a multi-pulse state is synchronized (but not completely) due to repulsion between the pulses. At a large overcriticality, the multi-pulse state goes over into a turbulent one. Lastly, breathers do not exist on top of the small-amplitude background.

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## FIGURE CAPTIONS

Fig. 1. The neutral stability curve  $\gamma_0(q)$  at  $\sigma = 1$ ,  $\gamma_1 = 0.9$ ,  $\gamma_2 = 5$ , and  $\Gamma = 1.35$ .

Fig. 2. The minimum value of the gain giving rise to instability of the zero background vs.  $\Gamma$  at  $\sigma = 1$ ,  $\gamma_1 = 0.9$ , and  $\gamma_2 = 5$ .

Fig. 3. A stable solitary pulse existing on top of the small-amplitude background at  $\gamma_0 = 0.54$  and  $\Gamma = 1.35$ : the wave field  $|u(x)|$  (a), and the time dependence of  $\text{Re } u$  at two fixed spatial points, one at the center of the pulse and one in the background (b).

Fig. 4. Snapshot of the wave field  $|u(x, t)|$  in the localized pulse existing on top of the chaotic background in the case  $\gamma_0 = 0.61$  and  $\Gamma = 1.35$  (a), and the time evolution of  $|u(x, t)|$  (b).

Fig. 5. Details of the random walk of the coordinate  $X_p(t)$  (a), and diffusive growth of the mean-square displacement  $\langle (X_p(t + \Delta t) - X_p(t))^2 \rangle$  (b) of the solitary pulse from fig. 4.

Fig. 6. Instantaneous snapshot (a) and the time evolution (b) of the wave field  $|u|$  at  $\gamma_0 = 0.61$  and  $\Gamma = 1.35$ . Synchronization of the random walk of three spontaneously formed pulses is seen.

Fig. 7. The same as in fig. 6(b), but for the system four times as long. As an initial condition, we use four copies of the snapshot pattern of fig. 6(a), that is,  $u(x, 0)$  for  $0 < x < 120$  takes the randomly perturbed value of fig. 6(a), and  $u(x, 0)$  for  $120 < x < 240$  also takes the value of fig. 6(a) perturbed randomly, and so on. So initially there are 12 pulses. It is clearly seen that the synchronization of the random walk of different pulses is imperfect.

Fig. 8. A “turbulent” configuration of  $|u(x, t)|$  at  $\gamma = 1.05$  (a) and  $\gamma = 2$  (b) for  $\Gamma = 1.35$ .

Fig. 9. A stable breather observed at  $\gamma_0 = 0.18$  and  $\Gamma = 0.2$ : the field  $|u(x, t)|$  (a), and its value at the central point of the breather vs. time (b).

Fig. 10. The amplitude  $A$  of the oscillations of the breather’s field at its central point (see fig. 9b) vs. the gain parameter  $\gamma_0$  (a), and  $A^2$  vs.  $\gamma_0$  near the critical point (b). The value  $\gamma_0 \approx 0.156$ , at which the oscillation amplitude vanishes, is a point of the transition (Hopf bifurcation) between the stationary pulse and breather.

Fig. 11. The phase diagram in the parametric plane  $(\Gamma, \gamma_0)$ , the other parameters taking the fixed values (6). The zero background is stable beneath the line marked by squares; localized pulses disappear

(decaying to zero) beneath the line marked by diamonds; crosses mark the border between the stable stationary solitary pulses (above the border) and stable breathers (below the border).

Fig. 12. A stable bound state of two breathers in the same case as in fig. 9. Two pulses are set up apart from each other as an initial condition.









































